21

RELATED RESULTS

It is finally appropriate to review a series of results which are related to the analysis of exact solutions describing the collision of plane waves in a flat background, which is the main subject of this book. The results described in this chapter differ from those given in previous chapters in a variety of ways. A number of interesting results have been obtained using different approaches, by considering the background in region I to be non-flat, or by considering alternative gravitational theories.

21.1 Other wave interactions

In this book we have considered specifically the collision and subsequent interaction of plane waves in a flat background. Initial conditions have been set up to consider this situation. There is, however, a considerable volume of literature in which wave interactions are considered in very different contexts.[1] Some very brief comments on some particular situations are now appropriate.

It may first be observed that the Gowdy cosmologies (Gowdy 1971) which have plane symmetry can be considered to represent closed universes built from interacting gravitational waves.[2] The similarity between these solutions and those for colliding plane waves has already been noted in Sections 10.7 and 17.3. In accordance with a cosmological interpretation, these solutions start with an initial 'big bang' singularity from which the waves emerge. They thus include the time reverse of the interaction region for some colliding plane waves.

A related situation has been considered by Hayward (1990*b*), who has considered plane symmetric space-times which are asymptotically flat in the past. These space-times are thus effectively the time reverse of some Gowdy cosmologies. This situation includes the collision of plane waves in a flat background as considered in previous chapters, but also includes solutions that are nowhere flat. The field equations are clearly identical to those considered above, but the initial conditions are different. For any general class of solutions, instead of imposing the colliding wave conditions described in Chapter 7, Hayward has simply imposed the condition that the space-time is asymptotically flat in the past. In terms of the familiar null coordinates u and v, the initial data are effectively those considered to be specified by initial asymptotic waves at past null infinity as $u \to -\infty$ and $v \to -\infty$. Hayward has given a particular solution in which the asymptotic waves have bounded amplitude, and has shown that interacting waves of this type generically produce a future curvature singularity.

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[1] For a review of studies on the interaction of gravitational waves with matter and fields, and their possible detection by these means, see Grishchuk and Polnarev (1980) and Sibgatullin (1984).

[2] See for example Centrella and Matzner (1979), Moncrief (1981), Carmeli, Charach and Malin (1981), Carr and Verdaguer (1983), Ibañez and Verdaguer (1983), Adams *et al.* (1982), Feinstein and Charach (1986) and Feinstein (1987).

21.2 Collisions in non-flat backgrounds

Apart from in the last section, all the colliding plane wave solutions that have been obtained have been considered to occur in a flat Minkowski background. This constitutes a very severe restriction that limits the application of the results obtained to real physical situations.

The collision of plane gravitational waves in a non-flat background has been considered by Centrella and Matzner (1979, 1982), and by Centrella (1980). This work was motivated by a consideration of situations in cosmology which involve the interaction of gravitational waves. The assumption of plane symmetry is imposed, so that the background region can be taken to be described by the Kasner metric. The approaching gravitational waves are then assumed to have the same plane symmetry. This situation is well posed and the interaction following the collision is uniquely determined.

In Chapter 5, it was described how the collision of initially non-expanding gravitational waves results in the astigmatic focusing of each wave after the collision. Such focussing will also occur in this case. However, if the background space-time is assumed to be expanding, the approaching gravitational waves may also be assumed to be expanding. In this case, as the two waves pass through each other, their rates of expansion will be reduced through the introduction of shear, but the waves will not necessarily start to contract. It may thus intuitively be expected that expanding gravitational waves will continue to expand after a collision, although with a slightly reduced rate of expansion. In this case, a singularity will be expected to occur in the past prior to the collision, and a future singularity will not necessarily be expected.

These properties are all confirmed by the detailed calculations of Centrella and Matzner (1979, 1982) using a Green's-function analysis, and also by Centrella (1980) using direct numerical techniques. Initial conditions are set on a time-like slice through three initial regions with different Kasner metrics being used in each region. Impulsive gravitational waves are generated from the joins in the initial data, and these subsequently collide and interact. Numerical integration indicates the expected properties. An explicit analytic solution, however, has not yet been obtained.

21.3 Solitons

Solitons occur in a number of physical theories. They arise as a feature of certain types of non-linear differential equations. Gravitational solitons appear quite naturally in the general theory of relativity, and their properties have been extensively investigated, particularly in a cosmological context.

Gravitational solitons are localized perturbations of the gravitational field which propagate on a homogeneous background. They have no dispersion. It has been suggested that their interaction in a cosmological context could play an important role in the process of isotropization in the early universe.

Mathematically, exact solutions describing solitonic behaviour can be generated using the inverse scattering technique of Belinskii and Zakharov (1978) from an initial 'seed' solution. This technique can be applied whenever the space-time has two commuting Killing vectors. Such solutions can always be interpreted as having plane symmetry. It may be recalled that the Ferrari–Ibañez solution for colliding gravitational waves that was described in Section 10.4 was originally obtained using this technique.

The collision and interaction of solitons in general relativity was initially investigated by Ibañez and Verdaguer (1983). In this paper, they took the homogeneous Kasner solution as the 'seed' metric. This describes an expanding vacuum cosmology with an initial singularity. In this case, the Belinskii–Zakharov inverse scattering technique generates pairs of solitons that are identified at the initial singularity and then propagate in opposite directions. By generating two such pairs at different locations, it is possible to consider the collision and interaction of two solitons, one coming from each pair.

The solution given by Ibañez and Verdaguer (1983) has a diagonal metric, and can therefore be considered as describing the collision and interaction of solitons with constant aligned polarization. In this solution, the line element is taken in the form of (10.59), namely:

$$\mathrm{d}s^2 = \frac{e^{-S}}{2\sqrt{\tilde{t}}}\left(\mathrm{d}\tilde{t}^2 - \mathrm{d}\tilde{z}^2\right) - \tilde{t}\left(e^{V}\mathrm{d}x^2 + e^{-V}\mathrm{d}y^2\right). \tag{21.1}$$

The exact solution is given by

$$\begin{aligned} e^V &= \tilde{t}^{\delta}\sigma_1\sigma_2 \\ e^{-S} &= \frac{2\tilde{t}^{(\delta^2/2-8)}{\sigma_1}^{\delta}{\sigma_2}^{\delta}}{H_1H_2(1-\sigma_1)^2(1-\sigma_2)^2}\left\{\left[(\sigma_1+\sigma_2)\tilde{t}^2-\frac{8\tilde{z}_1\tilde{z}_2\sigma_1\sigma_2}{(1+\sigma_1)(1+\sigma_2)}\right]^2\right. \\ &\qquad\qquad\qquad\qquad\left.-\frac{64{\omega_1}^2{\omega_2}^2{\sigma_1}^2{\sigma_2}^2}{(1-\sigma_1)^2(1-\sigma_2)^2}\right\}^2 \end{aligned} \tag{21.2}$$

where δ is the parameter of the seed Kasner metric and, for $i = 1, 2$,

$$H_i = (1-\sigma_i)^2 + \frac{16{\omega_i}^2{\sigma_i}^2}{(1-\sigma_i)^2\tilde{t}^2}, \qquad \tilde{z}_i = \tilde{z}_i^{\circ} - \tilde{z}. \tag{21.3}$$

The functions $\sigma_i(\tilde{t}, \tilde{z})$ are obtained from the complex ‘pole trajectory’ equation

$${\mu_i}^2 - 2(\tilde{z}_i - i\omega_i)\mu_i + \tilde{t}^2 = 0, \qquad \sigma_i = \mu_i\bar{\mu}_i/\tilde{t}^2 \tag{21.4}$$

and the ‘soliton’ parameters $\tilde{z}_i^{\circ}$ and ω_i are arbitrary real constants which indicate the ‘origins’ and ‘widths’ of the two pairs of solitons.

When seeking to interpret this solution it may be noted that, at the initial time $\tilde{t} = 0$, one pair of solitons is localized at $\tilde{z}_1^{\circ}$ and the other pair at $\tilde{z}_2^{\circ}$. For small values of ${\omega_i}^2 \ll 1$ the solitons are very localized. By analysing the structure of the curvature tensor, it can be shown that these solitons are gravitational fields with both radiative and non-radiative components. They can be interpreted as quasiparticles which evolve towards pure gravitational waves.

In each pair, the solitons propagate in opposite $\tilde{z}$ directions. The approaching solitons collide and pass through each other, forming the same type of interaction region that we have been considering for colliding plane waves. In this case, however, the interaction region tends asymptotically towards the Kasner background. A singularity does not develop. This result is not surprising as the solitons collide in an expanding background and the singularity is in the past at $\tilde{t} = 0$. The null vectors tangent to the solitons continue to expand after the collision, but the expansion is reduced by the familiar ‘focusing effect’.

An exact solution describing a similar situation in which a soliton collides with a plane gravitational wave has been obtained by Céspedes and Verdaguer (1987). In this case, the initial gravitational wave is taken to be one of the class of inhomogeneous non-diagonal solutions of Wainwright (1979) that can be interpreted as a gravitational wave pulse propagating on a Kasner background. It is appropriate to restrict the choice of the arbitrary function of a null coordinate to start with so that the simplest solution for a gravitational wave on a Kasner background can be obtained. This is used as a seed solution in a soliton transformation with four complex pole trajectories. The resulting metric is very complicated, but can be interpreted by considering certain asymptotic regions.

As indicated above, this technique generates solutions in which a pair of solitons are created at the initial singularity and propagate in opposite directions. In this solution one of the solitons subsequently collides with the gravitational wave pulse of the seed solution. Céspedes and Verdaguer (1987) have analysed the asymptotic structure of this solution, and have shown that the two solitons ultimately differ due to the interaction of one with the pulse wave. The soliton and the gravitational wave both continue to expand after the collision and there is a singularity in the past, but mutual focusing effects can be seen. In particular, they have demonstrated that the gravitational pulse wave becomes stronger and more polarized after the collision.

The exact solutions so far described in this section have both been vacuum solutions. The above solutions of Ibañez and Verdaguer (1983) have been extended to soliton collisions in cosmologies with matter by Cruzgate *et al.* (1988). The technique they have developed starts with a solution of the Einstein field equations coupled to a massless scalar field in a five-dimensional space-time with three commuting Killing vectors. Then, using the inverse scattering method of Belinskii and Zakharov (1978) and a suitable Kaluza–Klein dimensional reduction procedure, they have obtained solutions of the four-dimensional Einstein equations for a baratropic perfect fluid source. This $4m$-soliton family of solutions with complex pole trajectories depends on two arbitrary parameters.

These solutions have an initial curvature singularity. The soliton collision is associated with a high degree of anisotropy and inhomogeneity, leading to the formation of voids and halos. At large times, due to the matter–soliton interactions, the space-time tends asymptotically towards the Friedmann–Robinson–Walker background. The expression for the energy-momentum tensor is very complicated and the conditions for realistic fluids impose such severe constraints on the parameters that no satisfactory explicit solutions have been presented. However, a number of the explicit solutions considered can be re-interpreted in terms of the Brans–Dicke theory.

21.4 Alternative gravitational theories

Throughout this work, the collision and interaction of plane waves has been considered only in terms of Einstein's general theory of relativity.

There are, however, many alternative theories of gravitation, and it may be of interest to investigate the collision and interaction of waves in these theories. So far very little work has been done along these lines. In fact the results of only one paper will be mentioned here.

Rosenbaum, Ryan, Urrutia and Matzner (1986) have obtained an exact solution describing the collision of plane waves in classical $N = 1$ supergravity. In contrast to the situation in general relativity, this solution is non-singular everywhere. The two waves do not mutually focus each other. Although the waves shear, the Grassman algebra requires that the square of the shear vanishes and so, according to equation (5.1a), the waves are not induced to focus.

There is a clear need for more work to be undertaken to investigate the collision and interaction of plane waves in other gravitational theories, including quantum supergravity, in which the work of Rosenbaum, Ryan, Urrutia and Matzner may appear as some sort of semiclassical limit.

21.5 Numerical techniques

The emphasis throughout this book has been on the derivation and analysis of exact solutions describing colliding plane waves. It must be pointed out, however, that numerical techniques for analysing this situation have also been developed.

For their work on wave interactions in plane symmetric cosmologies, Centrella and Matzner (1979) and Centrella (1980) have developed techniques for numerically solving the field equations given Cauchy data on an initial space-like hypersurface. Their techniques follow the Hamiltonian approach of a (3+1) splitting of space-time. Their method can deal with the impulsive gravitational waves that are generated from the joins in the initial data and their subsequent collision and interaction. Centrella and Matzner (1982) have specifically applied this method to colliding plane impulsive waves in an expanding background. In this case the difficulties which occur with the development of future singularities are avoided.

Powerful numerical methods for dealing with the the characteristic initial-value problem in general relativity have been developed by Stewart and Friedrich (1982). In this case constraint-free initial data are imposed on a space-like 2-surface and two null hypersurfaces containing it. This is clearly the type of the initial data for the colliding plane wave problem as discussed in this book.

The approach of Stewart and Friedrich is based on a (2+2) splitting of space-time using two null directions. It was subsequently applied by Corkill and Stewart (1983) to a number of situations in which there exist two Killing vectors. In particular, they have applied it to the numerical evolution of the Schwarzschild space-time and to the collision and interaction of certain plane gravitational waves in a flat background.

Although this approach has clear advantages over the Cauchy problem, there are a number of inherent difficulties. These arise particularly because of the singularities that develop in these situations. It is therefore necessary to work with families of hypersurfaces that do not intersect the singularity. Corkill and Stewart have been able to devise methods by which it is possible to deal effectively with the evolution equations right up to the curvature singularity. The results they have obtained for colliding impulsive gravitational waves are entirely consistent with the Khan–Penrose solution described in Chapter 3.